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JAN 07

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University College London
DEPARTMENT OF MATHEMATICS
Mid-Sessional Examinations 2007
Mathematics 1101
Friday 12 January 2007 11.30 – 1.30 or 12.15 – 2.15

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*

The use of an electronic calculator is **not** *permitted in this examination*

1. State what it means for a real sequence to converge.

   Show that if $\lim_{n\to\infty} x_n = A$ and $\lim_{n\to\infty} y_n = B$ then $\lim_{n\to\infty}(x_n + y_n) = A + B$.

   Prove that every real sequence has a monotone subsequence.

   State and prove the Bolzano–Weierstrass Theorem. (You may assume that bounded monotone sequences converge.)

2. Let a be a number greater than 1. Show that the function $\mathbf{R} \to \mathbf{R}$ given by

$$x \mapsto a^x$$

   is increasing: that is, that if $x < y$, then $a^x < a^y$.

   Now define a sequence inductively by

$$\begin{aligned} x_0 &= 0, \\ x_{n+1} &= a^{x_n}, \text{ for } n \geqslant 0. \end{aligned}$$

   Show that whatever the value of a, the sequence is monotone increasing.

   Show that if $a = \sqrt{2}$, the sequence is bounded above by 2, and deduce that it converges.

   You may assume that the equation $x = (\sqrt{2})^x$ has exactly two real solutions: what are they?

   What is

$$\lim_{n\to\infty} x_n?$$

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3. Prove that if $s > 1$ then

$$\sum_{n=1}^{\infty} \frac{1}{n^s} < \infty.$$

Deduce that the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$$

converges.

(You may assume standard convergence tests provided that you state them clearly.)

Show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

4. Use the series definition of e^x to show that for each real x

$$1 + x \leqslant e^x$$

and that for $x < 1$,

$$e^x \leqslant \frac{1}{1-x}.$$

Use the first of these inequalities to show that for every positive y and every natural number n,

$$n\left(y^{1/n} - 1\right) \geqslant \log y.$$

Use the second inequality to show that if y is positive and n is sufficiently large,

$$n\left(y^{1/n} - 1\right) \leqslant \frac{\log y}{1 - 1/n \log y}.$$

Deduce that if y is positive,

$$\lim_{n \to \infty} n\left(y^{1/n} - 1\right) = \log y.$$

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5. *State and prove* the Cauchy–Schwarz inequality.

Let $(x_i)_1^n$ and $(y_i)_1^n$ be finite sequences of positive numbers satisfying

$$x_i \leqslant 2y_i \quad \text{and} \quad y_i \leqslant 2x_i$$

for each i. By considering the sum

$$\sum_{i=1}^{n}(2y_i - x_i)(2x_i - y_i),$$

show that

$$\sum x_i y_i \geqslant \frac{4}{5}\sqrt{\sum x_i^2}\sqrt{\sum y_i^2}.$$

6. *State* the Intermediate Value Theorem for functions $\mathbf{R} \to \mathbf{R}$.

Assuming properties of the exponential function (which you should state clearly) explain carefully how to construct the function

$$x \mapsto \log x.$$

Prove that $\log(xy) = \log x + \log y$ for positive x and y.

Prove that $x \mapsto \log x$ is continuous on $(0, \infty)$.

END OF PAPER